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Cyclotron Echo Phenomena*

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The physical mechanisms by which echoes are produced in a system of many charged particles gyrating in a static magnetic field is outlined. It is then shown how these ideas carry over to large systems of (almost) harmonic oscillators and are therefore applicable to many other physical systems. [*This is another in the series of review–tutorial papers that are written at the invitation of the Editor and that will be published from time to time in the American Journal of Physics. Articles in this series will be of an expository, critical nature and are intended to be eminently readable by students and professionals alike as a first introduction to a specialized topic.*]

INTRODUCTION

The existence of *echoes* in systems of precessing spins has been known for quite some time[1] and studied extensively. Hahn[1] and Purcell[2] have presented a very simple and elegant way of describing the spin echo formation in a rotating coordinate system. When cyclotron echoes in plasmas were discovered experimentally,[3] it soon became evident that the nice physical picture of spin echo formation could not be carried over directly to describe the cyclotron echo. Despite the formal similarities between the two systems, there is a very important difference: The response of a gyrating charged particle to a resonant radio-frequency *electric* field is, to a very high degree, linear in the electric field; whereas, the response of the precessing spin to an applied resonant radio-frequency *magnetic* field is not linear in the applied field. This difference will be brought out more clearly in material which follows, as well as the basic importance of the *nonlinear* behavior of the system.

Because the nonlinearity in a system of gyrating charged particles is very small, we first analyze their linear behavior and find that, although echoes do not occur in this approximation, there is a remarkable bunching in phase space, and the existence of small nonlinearities of various types can lead to echoes.[4,5] Because of the formal similarity of the linearized equation of motion of a charged particle in a static magnetic field with the harmonic oscillator equation, the ideas developed here have a much wider applicability than might otherwise appear. We elaborate on this point in a later section and outline the general conditions which are sufficient for echo production in other *physical systems*.

A brief summary of experimental results is in order. Echo experiments to date deal with *electron*-cyclotron resonance, which for laboratory magnetic fields of 1–3 kG, requires microwave frequencies of 3–9 GHz. Figure 1 indicates schematically the physical arrangement of the plasma,

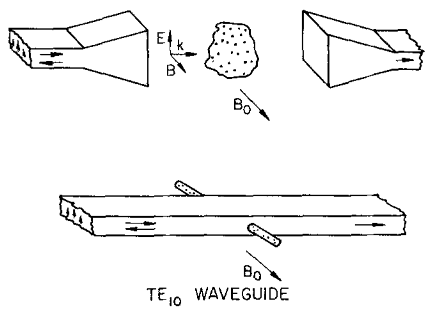

FIG. 1. Schematic arrangement for cyclotron echo experiments. The static magnetic field is mutually perpendicular to the electric field of the pulses and to their direction of propagation.

* Based on an invited talk before the American Physical Society, Bull. Amer. Phys. Soc. **10,** 1191 (1965); see also Ref. 5.

[1] E. L. Hahn, Phys. Rev. **80,** 580 (1950).

[2] H. Y. Carr and E. M. Purcell, Phys. Rev. **94,** 630 (1954).

[3] R. M. Hill and D. E. Kaplan, Phys. Rev. Letters **14,** 1062 (1965).

[4] R. W. Gould, Phys. Letters **19,** 477 (1965).

[5] R. W. Gould, "Cyclotron Echo Phenomena," Astia Document AD 627–681 (Dec. 1965).

magnetic field, and radio-frequency electric field which are frequently used. Of importance here are the facts that the resonant electric field is perpendicular to the static magnetic field $\mathbf{B}_0$, and that the dimension of the plasma in the direction of wave propagation is small so that each plasma electron is accelerated with a radio-frequency field of approximately the same amplitude and phase. Figure 2 shows the magnitude of the applied radio-

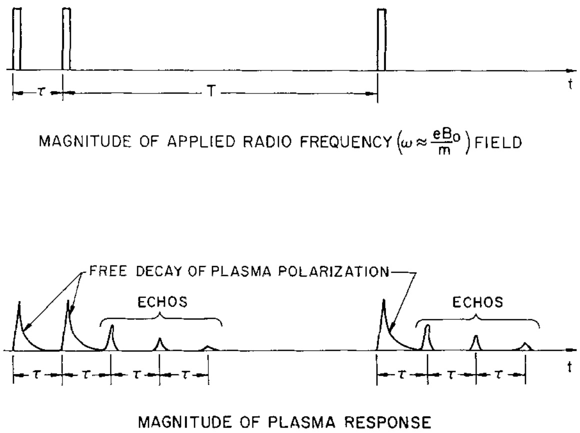

FIG. 2. Sequence of applied pulses and plasma response.

frequency field and the corresponding response (electrical current) produced in the plasma. The radio-frequency field has a frequency ω approximately equal to the electron cyclotron gyration frequency $\omega_c = eB_0/m$ and is in the form of short sinusoidal bursts or pulses consisting of 30–100 cycles. The response to two pulses (*two-pulse echoes*) as well as the response to three pulses (*three-pulse echoes*) is indicated. In the case of two-pulse echoes we see, in addition to the direct response of the plasma to the two applied pulses, several delayed responses (echoes) at intervals equal to the spacing between the two applied pulses τ. We expect, in the case of a linear system, that the response to two pulses would simply be the superposition of the individual responses to the two pulses, and hence there would be no echoes. From the appearance of the delayed responses which cannot be constructed by superposition, we conclude that the echoes involve, in an essential way, some nonlinear behavior of the system. The application of a third pulse still much later ($T \gg \tau$) results in not only a direct response of the plasma to that pulse, but to another series of three-pulse echoes with separation τ equal to the interval between the first two pulses. A very surprising aspect of this result is that the interval T can be so long, that during this time the electrons suffer *many collisions* with neutral atoms of the plasma, yet still "remember" enough to produce echoes with the original two-pulse interval τ. Under such conditions *two-pulse* echoes with the same large interval T are completely destroyed by the collisions.

Finally, we note that cyclotron echo experiments, to date, have been performed in the afterglow of a rare gas (argon or neon, frequently) decaying plasmas,[6–8] or in steady state Q-machine plasmas[9] (cesium vapor). These plasmas generally, although not always, have the following features: low electron density ($n_e \sim 10^8$–$10^{10}\,\mathrm{cm}^{-3}$), low electron energy ($kT_e \sim 0.05$–$0.5$ eV), low collision frequency for electron momentum transfer ($\nu \sim 10^6$–$10^7\,\mathrm{sec}^{-1}$) and small linear dimensions (compared to the wavelength of the incident radiation), and relatively homogeneous static magnetic field $\Delta B_0/B_0 \sim 0.01$–0.001. In addition, electrons typically receive energies of 1–10 eV from the rf pulses.

I. LINEAR BEHAVIOR OF A SYSTEM OF INDEPENDENT PARTICLES

The experimental conditions just outlined suggest a number of simplifying assumptions in the analytic treatment of the gyrating, charged-particle behavior:

(a) Each particle moves independently of the others in the external fields, i.e., neglect collective effects;

(b) neglect the initial velocity of an individual electron, and hence its motion along magnetic-field lines;

(c) each particle in the plasma experiences the *same* rf field, and we compute the total induced plasma current by simply adding the contributions from individual particles;

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[6] G. F. Herrmann, R. M. Hill, and D. E. Kaplan, Phys. Rev. **156**, 118 (1967).

[7] R. S. Harp, R. L. Bruce, and F. W. Crawford, J. Appl. Phys. **38**, 3385 (1967).

[8] L. O. Bauer, F. A. Blum, and R. W. Gould, Phys. Rev. Letters **80**, 435 (1968).

[9] D. E. Kaplan, R. M. Hill, and A. Y. Wong, Phys. Letters **22**, 585 (1966).

(d) neglect collisions in case of two-pulse echoes ($\nu\tau \ll 1$) and assume complete isotropic scattering in case of three-pulse echoes ($\nu T \gg 1$);

(e) use of linearized equation of motion for individual electron (i.e., neglect relativistic effects, neglect rf magnetic field, neglect spatial variation of electric field over electron orbit);

(f) because of the slight spatial inhomogeneity of the static magnetic field, electrons in different parts of the volume have slightly different cyclotron frequencies.

At appropriate points in the discussion that follows, we shall remark more fully on implications of these assumptions, and the sometimes important consequences of taking into account the neglected effects. The following approximate analysis also forms an effective framework upon which to treat echo phenomena in almost-linear systems.

The nonrelativistic equation of motion for a single electron is

$$d\mathbf{v}/dt = -(e/m)(\mathbf{E}+\mathbf{v}\times\mathbf{B}), \tag{1}$$

where $\mathbf{E}$ and $\mathbf{B}$ are the electromagnetic fields at the particle. The magnetic field consists of the static field $\mathbf{B}_0$ and the rf magnetic field. The effect of the latter on the electron motion is v/c times smaller than the rf electric field $\mathbf{E}$ and will be neglected. Furthermore, we neglect the spatial dependence of $\mathbf{E}$ and $\mathbf{B}_0$ over the orbit and take as our linearized equation of motion

$$(d\mathbf{v}/dt)+\boldsymbol{\omega}_c\times\mathbf{v} = -(e/m)\mathbf{E} \qquad \boldsymbol{\omega}_c \equiv (e/m)\mathbf{B}_0, \tag{2}$$

where $\boldsymbol{\omega}_c$ is the (vector) cyclotron frequency of the particle under consideration, and $\mathbf{E}$ is applied field (the same for all particles of the plasma). As in the case of magnetic resonance phenomena we find it very convenient to use a rotating coordinate system. When we resolve the oscillating electric field into two counter-rotating constant electric field vectors, one of the rotating components will rotate with the same sense, and nearly the same rate if $\omega \simeq \omega_c$, as the natural gyration of the electron in a magnetic field. Only this (resonant) component produces a significant motion of electron, and the other may be neglected provided the rf pulse lasts for many cycles. We chose the rotating coordinate system so that the rotation axis is parallel to $\mathbf{B}_0$, and so that the resonant component of $\mathbf{E}$ appears stationary (and is perpendicular to $\mathbf{B}_0$). Thus in the rotating coordinate system the equation of motion is

$$(d\mathbf{v}'/dt)+\boldsymbol{\omega}_c'\times\mathbf{v}' = -(e/m)\mathbf{E}' \qquad \boldsymbol{\omega}_c' = \boldsymbol{\omega}_c - \boldsymbol{\omega}, \tag{3}$$

where $\mathbf{E}'$ stands for the static electric field seen in the rotating system ($E'=E/2$), and ω_c' is the *difference* between the cyclotron frequency and the applied frequency and is very much smaller than either ω_c or ω since $\omega \simeq \omega_c'$. Since the motion is confined to the x, y plane, to further simplify the mathematical manipulations we introduce complex scalar quantities $\hat{v}'=v_x'+iv_y'$, and $\hat{E}'=E'_x+iE_y'$ in place of the real vector quantities. The real part of a complex quantity gives the x component of the vector. In this notation the perpendicular part of Eq. (3) may be written

$$(d\hat{v}'/dt)-i\omega_c'\hat{v}' = -(e/m)\hat{E}'. \tag{4}$$

The solution of this ordinary linear differential equation may be written

$$\hat{v}'(t) = \hat{v}'(0)\exp[i\omega_c'(t-t_0)] - \frac{e}{m}\int_{t_0}^{t}\hat{E}'(s)\exp[i\omega_c'(t-s)]ds, \tag{5}$$

where the first term is the solution of the homogeneous (field free $E'=0$) equation and the second term describes the additional effect produced by the resonant electric field. For a rectangular pulse $[\hat{E}'=\hat{E}_1', -t_1/2<t<t_1/2]$. Equation (5) may be integrated at once to give

$$\hat{v}'(t) = \left\{v'(0) - \frac{e}{m}\hat{E}_1't_1\left[\frac{\sin\frac{1}{2}(\omega_c't_1)}{\frac{1}{2}(\omega_c't_1)}\right]\right\}\exp(i\omega_c't) \tag{6a}$$

$$\cong[\hat{v}'(0)-(e/m)\hat{E}_1't_1]\exp(i\omega_c't), \qquad \omega_c't_1\ll 1. \tag{6b}$$

We see that the effect of the resonant rf-field pulse is simply to add to the complex velocity before the pulse. The latter form of Eq. (6b) holds for pulses sufficiently short ($\omega_c't_1 \ll 1$) that all electrons acquire the same increment of velocity $\hat{v}_1 = -(e/m)E_1't_1$, regardless of the difference cyclotron frequency ω_c'. Henceforth we shall simply speak of the (complex) velocity imparted to each electron by the pulses ($\hat{v}_1$, $\hat{v}_2$, and $\hat{v}_3$, respectively).

This result can now be used to discuss the motion of a single electron, or a system of many noninteracting electrons when subjected to two pulses. Since the actual physical position of the electron is unimportant, except insofar as this determines its cyclotron frequency ω_c and hence its difference cyclotron frequency $\omega_c'=\omega_c-\omega$, we trace out its velocity in the rotating coordinate system. Figure 3 shows the velocity configuration

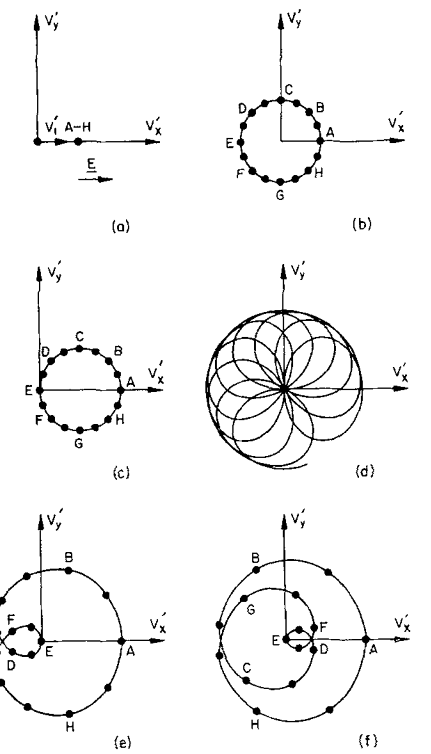

FIG. 3. Velocity space configuration for a system of particles at various instants; (a) $t=0^+$, (b) $t=\tau^-$, (c) $t=\tau^+$, (d) $t=\sim 1.5\tau$, (e) $t=2\tau$, (f) $t=3\tau$.

of the system at various instants of time. Prior to the first pulse all electrons are at the origin, since we neglect initial velocities. Just after the first pulse (3a), all electrons have acquired the same velocity $\hat{v}_1$ which we take, for simplicity, to be in the x' direction. Subsequent to the first pulse, according to Eq. (6b), each electron executes a slow circular orbit with angular velocity ω_c'. Since there is a range of difference cyclotron frequencies related to spatial inhomogeneity of the magnetic field and the spatial distribution of electrons, let the number of electrons with difference cyclotron frequency ω_c' be denoted by $NG(\omega_c')d\omega_c'$. N is the total number of electrons in the system and $G(\omega_c')d\omega_c'$ is the *fraction* of the electrons with cyclotron frequency between ω_c' and $\omega_c'+d\omega_c'$ (note that

$$\int_{-\infty}^{\infty} G(\omega_c')d\omega_c'=1).$$

Let the frequency width of this distribution be characterized by $\langle\Delta\omega_c'^2\rangle^{1/2}$. After a long time $[t\gg 2\pi/(\langle\Delta\omega_c'^2\rangle)^{1/2}]$ electrons will be found at each point around the circle (3b) according to their difference cyclotron frequency, more or less equally distributed. The macroscopic current induced in the plasma by the pulse has decayed due to the loss of phase coherence of the particle motions (phase mixing). We calculate the plasma current by summing over particles, or, more precisely, integrating over the distribution of difference cyclotron frequencies

$$\hat{J}'(t)=-Ne\int \hat{v}'(t,\omega_c')G(\omega_c')d\omega_c', \quad (7a)$$

$$=-Ne\hat{v}_1 g(t), \quad (7b)$$

where

$$g(t)=\int G(\omega_c')\exp(i\omega_c't)d\omega_c'. \quad (8)$$

We see that initially the current is simply $\hat{J}=-Ne\hat{v}_1$, since $g(0)=\int G(\omega_c')d\omega_c'=1$. The function $g(t)$ describes the subsequent decay of the current which occurs due to the phase mixing of the various frequency components. $g(t)$ is essentially the normalized impulse response of the plasma and is simply the Fourier transform of the distribution of difference cyclotron frequencies. Thus the wider the range of cyclotron frequencies, the faster the plasma current decays due to phase mixing $(t_{\text{decay}}\sim 1\langle\Delta\omega_c'^2\rangle^{1/2})$. We note also, in terms of Fig. 3, the integral Eq. (7) amounts to a calculation of the *center of velocity* of the distribution of electrons in velocity space, with the real and imaginary parts giving the x and y components, respectively. Thus when the particles are equally distributed around the circle, the plasma current $\hat{J}$ vanishes.

Figure 3(b) shows the velocity configuration just prior to the second pulse $(t=\tau^-)$. Electrons whose velocity vector has rotated either through an angle $\omega_c'\tau=\pi/4$, or $\pi/4$ plus (or minus) some multiple of 2π, are together at B at this instant. Similarly, particles at C, or D, or E have undergone rotations of $\pi/2\pm2\pi n$, or $3\pi/4\pm2\pi n$, where n is some integer. Because of the wide range of angles, the center of velocity of Fig. 3(b) is at the origin, and the current vanishes. The second pulse simply adds a complex velocity $\hat{v}_2$ to the velocity of each electron according to Eq. (6b), whereupon it resumes its circular orbit around the origin from its *new* initial point and with its original angular velocity. The configuration just following the second pulse ($t=\tau^+$) is shown in Fig. 3(c) for the case $\hat{v}_2=\hat{v}_1$. We note the center of velocity is now at $\hat{v}_2$, corresponding to the plasma current $-Ne\hat{v}_2$ due to the second pulse. The electrons at B, for example, then disperse and continue in circular orbits about the origin with their respective angular velocities $[\omega_c'=(\pi/4\tau)\pm(2\pi n/\tau)]$ simply because they had reached B by circling the origin a different number of times. Thus, Fig. 3(c) breaks into a multileaf curve such as shown in Fig. 3(d), and the current due to the second pulse then decays, just as the current from the first pulse decayed.

Although most of the time the velocity configuration has the general appearance shown in Fig. 3(d), a very remarkable regrouping of electrons occurs at special times, $t=2\tau$, 3τ, $\cdots$, as can be seen by the following argument. Electrons which, in the first interval $0<t<\tau$ reached B, did so by gyrating through angles $\pi/4$ or $\pi/4+2\pi n$ (n=integer), and each will gyrate through precisely the same angle in the second interval $\tau<t<2\tau$, since its cyclotron frequency is assumed to be exactly the same during the second interval. Thus all particles which were together at b [in Fig. 3(c)] when $t=\tau$ will be together again at B [in Fig. 3(e)] when $t=2\tau$. Their position, shown in Fig. 3(e), has been rotated by $\pi/4$ from that in Fig. 3(c). Thus the multileaf curve of Fig. 3(d) coalesces into a single leaf shown in Fig. 3(e) [or into that shown in Fig. 3(f) when $t=3\tau$].

Despite this remarkable regrouping at $t=2\tau$, 3τ, $\cdots$, the plasma current is still essentially zero at these times, and no echo occurs in this approximation. This is because the centers of velocity of Figs. 3(e) and 3(f) are still at the origin, even though in Fig. 3(e), for example, there are more electrons with $v_x'<0$. Their contribution is exactly cancelled by the *fewer* electrons with *larger* positive v_x'. Thus we are led to the idea that despite the high degree of order in velocity space at specific times, the nonoccurrence of echoes in a linear system is associated with the *exact cancellation* in the mean velocity, and that effects which destroy the assumed linearity of the system, may spoil this exact cancellation and therefore give rise to a macroscopic plasma current and hence echoes. It is evident that effects which tend to distort the curve and hence its symmetry in Fig. 3(e) ($t=2\tau$) will result in a nonzero center of velocity, whereas in Fig. 3(d) ($\tau<t<2\tau$) effects which distort a single leaf of the curve will not, in general, produce a nonzero center of the velocity. Thus current is expected to appear in the plasma when $t=2\tau, 3\tau \cdots$).

II. NONLINEAR EFFECTS

We note that an important quantity which characterizes the *difference* between electrons at various points in Fig. 3, aside from their cyclotron frequencies or orbital phase, is their *energy* or *speed*. In our linear treatment we have implicitly assumed certain processes or factors to be independent of energy (or speed) which, if they were slightly dependent on energy (or speed), would provide the nonlinearity necessary for echo production.[4,10–13] These are,

(a) The effect of the electric field force in accelerating the electron,

(b) the frequency of electron gyration in the magnetic field,

(c) the probability of a phase-destroying collision with a neutral gas atom or molecule.

While we have so far ignored the effect of collisions of the electron with the neutral particles on the cyclotron echo, the nonlinear effect of energy-dependent elastic collisions is perhaps the simplest to understand in terms of Fig. 3. If we assume that the scattering by the heavy neutral particles is elastic and isotropic, then a scattering event simply displaces a particle to some other point on

[10] W. H. Kegel and R. W. Gould, Phys. Letters **19**, 531 (1965).

[11] G. F. Herrmann and R. F. Whitmer, Phys. Rev. **143**, 122 (1966).

[12] F. W. Crawford and R. S. Harp, J. Appl. Phys. **37**, 4405 (1966).

[13] R. M. White, J. Appl. Phys. **37**, 1693 (1966).

a sphere in velocity space with equal probability. On the average, one collision is sufficient to destroy the phase correlation of a particle's orbital motion with an rf pulse, and therefore remove it from further consideration insofar as echo production is concerned. If the probability of such a collision depends on the electron's energy, then the number of electrons surviving until $t=2\tau$ without making a collision (and thus contributing to the echo) will depend on whether it was *accelerated* by the second pulse (as in A, B, H in Fig. 3), or *decelerated* by the second pulse (as in D, E, F in Fig. 3). If the probability of a phase-destroying collision increases with electron energy, then relatively fewer particles will remain near H, A, B, than D, E, F [Figs. 3(e) or 3(f)], and the center of velocity will be shifted away from the origin, to the left, thus producing an echo pulse with a phase π relative to the responses to the applied pulses. Similarly if the collision probability decreases with energy, there are relatively *more* particles near HAB than near DEF, and the phase of the echo pulse is reversed. We discuss this process quantitatively later.

Should the frequency of cyclotron motion ω_c' be weakly dependent upon electron energy (anharmonic oscillator) as, for example, due to the relativistic increase in mass with energy, then the angle through which particles rotate during the second interval τ will be slightly different from the angle through which they rotated during the first interval τ in a manner which depends on whether they are near HAB near DEF. When the frequency decreases with energy, electrons at A are rotating more slowly and are thus displaced slightly downward, whereas electrons at E are unaffected. The symmetry with respect to the v_x' axis in Figs. 3(e) and 3(f) is thus destroyed and an echo with phase $-\pi/2$ is produced. It is important to note that, since it is actually the change in *phase* of the electron's orbital motion during the time τ, a very weak dependence on cyclotron *frequency* with energy may produce an appreciable phase change during the many cycles which elapse. Even so, the relativistic mass change of a 10 eV electron ($\Delta m/m \sim 10^{-5}$) is insufficient to explain the large echoes which are observed. Spatially nonuniform static electric and magnetic fields, in addition to causing particle drifts, also cause the frequency of the periodic cyclotron motion to be energy-dependent.[14,15] For example, a rotationally symmetric magnetic field whose strength decreases with radius, leads to a cyclotron frequency which decreases with the orbit radius of the particle and hence with increasing energy. The effect of nonuniform static electric fields, more important for the experiments, is described in the last section.

Spatial inhomogeneities in the rf applied field (both E and B) can cause the effect of these fields to be dependent on the electron energy through the dependence on orbit size.[11] When the orbit radius is not negligible compared to the wavelength of the plane-wave applied fields, the *effectiveness* of these driving fields decreases with increasing orbit size. Electrons at A [in Fig. 3(c)] will have been translated to the right slightly less than those at B because of their larger orbit radius during the second pulse. This results in a curve which is no longer a circle, but elliptical. The curve in Fig. 3(e) is, therefore, distorted in a corresponding manner with particles at A contributing slightly less than their share to the current, and the exact cancellation previously described is spoiled and an echo results. The size of this effect is actually very small, however, being of order (cyclotron radius/wavelength)$^2 \sim v^2/c^2$ and, unlike the relativistic frequency change, does not enjoy a multiplication by the number of cycles between the pulse. Thus this effect is unimportant for cyclotron echoes, although in other systems (spin systems) an energy-dependent driving force can be entirely responsible for echo formation (see Sec. V).

We now present a quantitative treatment of the echo formation process just described, when energy-dependent collisions and/or energy-dependent cyclotron frequencies are the responsible nonlinear mechanism. The following description is valid in the rotating system for an electron whose difference-cyclotron frequency is ω_c'. Just after the *first* pulse, the complex velocity is $\hat{v}=\hat{v}_1$. Just prior to the *second* pulse the velocity is $\hat{v}=\hat{v}_1 \exp(i\omega_c''\tau)$, and immediately following the second pulse the velocity is $\hat{v}=\hat{v}_1 \exp(i\omega_c''\tau)+v_2$, where $\omega_c'' \cong \omega_c' + (\partial\omega_c'/\partial v^2)\,|v_1|^2$ is frequently

[14] J. L. Hirschfield and J. M. Wachtel, Bull. Amer. Phys. Soc. **11**, 538 (1966).

[15] L. O. Bauer, R. W. Gould, and W. H. Kegel, Bull. Amer. Phys. Soc. **12**, 756 (1967).

following the first pulse. Following the second pulse the frequency is

$$\omega_c''' = \omega_c' + (\partial\omega_c'/\partial v^2)[\,|v_1|^2 + |v_2|^2 + 2|v_1||v_2|\cos(\omega_c''\tau+\phi)], \quad (9)$$

where ϕ is the difference in phase between $\hat{v}_1$ and $\hat{v}_2$, and the velocity is

$$v = [\hat{v}_1 \exp(i\omega_c''\tau) + \hat{v}_2] \exp(i\omega_c'''t),$$

where time is measured from the *second* pulse. Introducing $P(t)$ as the probability that an electron survives until time t, *without* making a phase-destroying collision as a factor, into Eq. (7) in computing the complex current, we obtain

$$\hat{J}(t) = -Ne \int [\hat{v}_1 \exp(i\omega_c''\tau) + \hat{v}_2] \times \exp(i\omega_c'''t) P(t) G(\omega_c') d\omega_c'. \quad (10)$$

Now $P(t) \sim e^{-\nu t}$ where $\nu = \nu(v^2)$ is the energy-dependent electron collision frequency. For simplicity in further discussion, we assume that $\nu(v^2) = av^2$, i.e. that the electron mean free path decreases inversely with v. This is a very good approximation for neon, but rather poor for argon. Furthermore, we can assume without loss of generality, that $\hat{v}_1$ and $\hat{v}_2$ are real, and hence that $\phi = 0$. Thus,

$$P(t) = \exp(-av_1^2\tau) \exp\{-a[v_1^2+v_2^2+2v_1v_2 \times \cos(\omega_c''\tau)]t\}, \quad (11)$$

where the first factor is the probability of no collisions between the first and second pulses, and the second factor is the probability of no collision between the second pulse and time t. We note that $\cos\omega_c''\tau$ appears in *two* exponentials, and expand this periodic function of $\omega_c''\tau$ in a Fourier series

$$\exp[(i\beta-\alpha)\cos\omega_c''\tau] = \sum_{n=-\infty}^{\infty} (-i)^n J_n(\beta+i\alpha) \exp(-in\omega_c''\tau), \quad (12)$$

where J_n is the Bessel function of order n and $\beta \equiv 2v_1v_2(\partial\omega_c'/\partial v^2)t$, and $\alpha = 2av_1v_2t$ are appropriate measures of the energy dependence of the gyration frequency and collision frequency, respectively. Thus $\alpha = 0$ if the responsible echo mechanism is energy-dependent gyration frequency, and $\beta = 0$ if the responsible echo mechanism is energy-dependent on the collision frequency. With this series representation, each of the integrals over ω_c' are of the type occurring previously, see Eq. (8), and the final result may be easily put in the following simple form:

$$\hat{J}(t) = -Ne \sum_{n=-\infty}^{\infty} \hat{V}_n(t) g(t-n\tau) \qquad t>0, \quad (13)$$

where $g(t)$ is given by Eq. (8) and $A_n(t)$ is the somewhat complicated expression

$$V_n(t) = (-i)^n \{\exp(a[v_1^2\tau + (v_1^2+v_2^2)t])\} \times \{\exp(ib[(v_1^2+v_2^2)t - nbv_1^2\tau])\} \times \{v_2J_n + iv_1J_{n+1}\}. \quad (14)$$

Equation (13) shows that the current consists of a regular train of similar pulses occurring at times $t \simeq n\tau$. Normally, the pulses' duration is short compared to the interpulse spacing (τ), and $A_n(t)$ does not vary appreciably during a pulse. Then the pulses do not overlap, and $V_n(n\tau)$ is the amplitude of the nth pulse. $n=0$ gives the response to the *second* applied pulse, and $n = 1, 2, \cdots$ corresponds to the first, second, $\cdots$ echo pulses. It is instructive to consider the two limiting cases, $a=0$ and $b=0$.

A. Energy-Dependent Frequency Only ($a=0$)

The first factor in curly brackets in Eq. (14) is unity. The second factor is simply a phase factor with magnitude unity, hence the echo amplitude is given by

$$|V_n| \cong [v_2^2 J_n(\beta) + v_1^2 J_{n+1}^2(\beta)]^{1/2}, \quad (15a)$$

$$\cong v_2(\beta/2)^n = v_2(v_1v_2bn\tau)^n/n! \qquad \beta^2 \ll 1 \quad (15b)$$

where the latter form holds for weak applied pulses. We note that successive echoes are weaker by a factor $\sim\beta/2 \ll 1$, but increase more rapidly with increasing pulse strength or interpulse separation $(A_n \sim v_1^n v_2^{n+1} \tau^n)$. The first echo is the largest in this weak echo limit, and its amplitude is proportional to the first pulse amplitude v_1 (and to the square of v_2). It is even possible for this echo to be larger than the first pulse; so that one may regard the first echo as an amplified and delayed response to the first pulse, with the second pulse v_2 serving the role of a trigger or pump. The ratio of plasma current in the first echo pulse to that produced by the first applied pulse is

$$|V_1/v_1| \cong bv_2^2\tau, \quad (15b)$$

which is numerically equal to the nonlinear phase change in the interval τ due to the second pulse alone. For amplification this must exceed unity. In Fig. 4 we show how the amplitude of the first

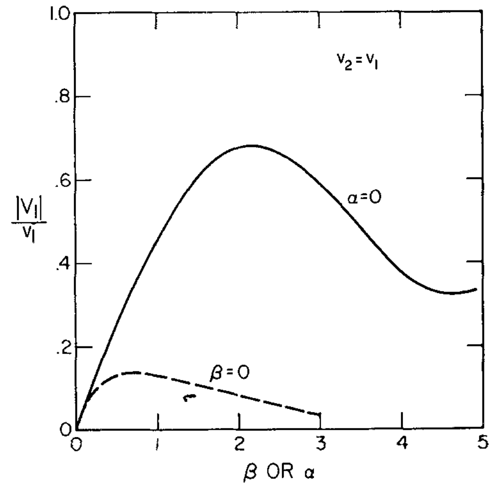

FIG. 4. Dependence of the first two pulse echo amplitudes on the nonlinear parameter β (solid curve, energy-dependent frequency), or α (dashed curve, energy-dependent collisions).

echo pulse depends on the applied pulse amplitudes (v_1, v_2) and interpulse space, as obtained from Eq. (15a).

B. Energy-Dependent Collisions Only ($b=0$)

In this case Eq. (14) becomes

$$V_n(n\tau) = \exp[-av_1^2\tau - a(v_1^2+v_2^2)n\tau] \times[v_2I_n(\alpha)+v_1I_{n+1}(\alpha)], \quad (16a)$$

$$= v_2(\alpha/2)^n/n! = v_2(v_1v_2an\tau)^n/n!,$$

$$\alpha/2 \ll 1 \text{ and } (v_2/v_1)\alpha \ll 1, \quad (16b)$$

where again, the latter form holds for weak applied pulses and successive echoes are weaker. At first glance amplification of the first echo appears possible from Eq. (16b) if $av_2^2\tau > 1$. However, in this case the exponential factor in Eq. (16a) is considerably smaller than unity, and it can readily be shown that for this collision law (and most reasonable collision laws) amplification is not feasible.

For other, more general collision laws of the form $\nu = \nu(v^2)$, the factor $P(t)$ is nevertheless a periodic function of $\omega_c''\tau$ through the dependence of v^2 on $\cos\omega_c''\tau$, and therefore can still be expanded in a Fourier series

$$P(t) = \exp[-\nu(v_1^2)\tau] \times \exp[-\nu(v_1^2+v_2^2+2v_1v_2\cos\omega_c''\tau)t],$$

$$= \sum_{-\infty}^{\infty} P_n(v_1, v_2, t) \exp(-in\omega_c''\tau). \quad (17)$$

The expansion coefficients P_n and P_{n+1} which enter the expression for the amplitude of the nth echo can be evaluated numerically, if necessary, for the collision law of interest using the usual formulas for the Fourier series coefficients.

When the effectiveness of the driving force is energy-dependent, the velocity increment produced by the second pulse will be dependent on $\cos\omega_c'\tau$ through a dependence on v^2. If the energy dependence is *weak*, the terms of a Taylor series expansion in v^2 (the constant and linear terms) suffice and we can write the velocity of a particle whose difference-cyclotron frequency is ω_c', at time t

$$v = [v_1\exp(i\omega_c'\tau) + (v_2 + d\cos\omega_c'\tau)]\exp(i\omega_c t), \quad (18)$$

where $d = 2v_1v_2(\partial v_2/\partial v^2)$. Performing the integration over difference cyclotron frequencies indicated in Eq. (7), we can put our result in the same form as Eq. (13) with

$$V_0 = v_2,$$
$$V_1(t) = d/2,$$
$$V_n(0) = 0 \qquad n > 1, \quad (19)$$

i.e., there is a single-echo pulse only.

III. THREE-PULSE ECHOES

A similar physical picture of the formation of three-pulse echoes can be given,[5] if it is assumed that an electron experiences very few collisions during the interval τ but many elastic collisions during the interval $T(T \gg \tau)$. The velocity space diagrams are shown in Fig. 5, where it is assumed that between the second and third pulses, each electron experiences many elastic collisions. Thus particles which were at B or H, for example, in

Fig. 5(c) will have been scattered onto a *spherical shell* in velocity space as in Fig. 5(d) (where we show only the intersection of this sphere with the $v_x'v_y'$ plane). Thus electrons are distributed with nearly equal probability per unit area, on different

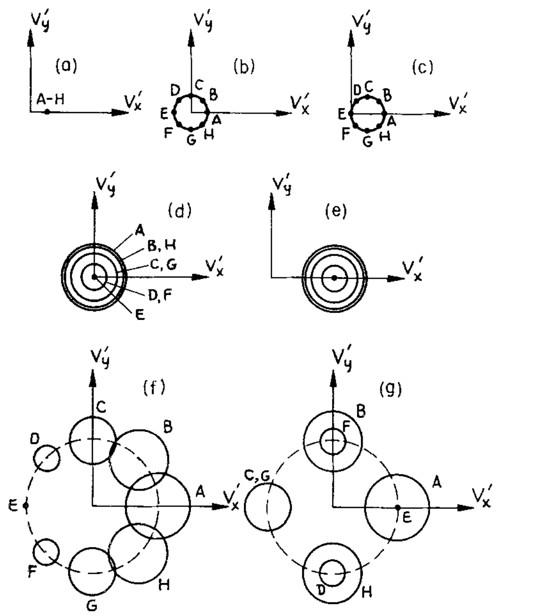

FIG. 5. Three-pulse echo space configuration for a system of particles at various instants; (a) $t=0^+$, (b) $t=\tau^-$, (c) $t=\tau^+$, (d) $t=(\tau+T)^-$, (e) $t=(\tau+T)^+$, (f) $t=2\tau+T$, (g) $t=3\tau+T$.

spherical shells in velocity space according to their values of $\omega_c'\tau$ just prior to the third pulse. Application of the third pulse simply translates each spherical shell producing the distribution shown in Fig. 5(e). Each electron then proceeds to execute a circular orbit about the v_z' axis at its original angular rate ω_c'. The shell B, H breaks into many leaves since the particles difference-cyclotron frequencies differ by multiples of $2\pi/\tau$. However, after an additional interval τ (or 2τ, 3τ, $\cdots$) these complicated surfaces coalesce into two surfaces, a cross section of which is shown in Fig. 5(f) [or Fig. 5(g) when the interval is 2τ]. The surface B, in Fig. 5(f), is just the surface B, in Fig. 5(e), but rotated through the angle $\omega_c'\tau$ for this group of particles.

If one were to calculate the plasma current at these instants of time using Eq. (7a), it would be essentially zero (i.e., no echo) despite the remarkable bunching in velocity space which has been demonstrated. One might say that, as in the case of the two-pulse echo, the three-pulse echo does not occur because of an exact cancellation in the contributions to $\hat{J}$; the effect of particles on shell B are canceled by those on shell F, those on shell A by those on shell E, etc. As we have already noted, three-pulse echoes are not expected to occur in *linear* systems. However, it is clear from our approach that any nonlinear effect which acts to spoil the exact cancellation will result in the occurrence of echoes.

It is important to note that a crucial assumption is implicit in this model: that the cyclotron frequency of a given particle is precisely the same during the interval τ following the third pulse, as it was during the original interval between the first and second pulses. This assumption, together with the fact that different energy shells in velocity space correspond to different values of $\omega_c'\tau$, allows the particles to regroup periodically in velocity space following the third pulse.

There are a large variety of nonlinear effects which may be responsible for three-pulse echoes. The various nonlinearities already enumerated for two-pulse echoes can act after the third pulse. Another class of effects acting during the interval between the second and third pulses can also occur. During this interval each electron experiences many collisions and in doing so may diffuse into a region where the difference-cyclotron frequency is no longer the same. As a result, such an electron does not contribute to the current in the manner just described. If this loss mechanism depends upon the energy of the electron, e.g., energetic electrons are lost more rapidly, then the cancellation argument no longer applies and three-pulse echoes result. At present it is not known which of the possible nonlinear mechanisms is responsible for three-pulse cyclotron echoes.

IV. GENERALIZATION TO OTHER SYSTEMS

Many physical systems can be represented by systems of harmonic (or almost harmonic) oscillators, having a distribution of oscillator frequencies, which are subjected to a common force and whose response consists of the sum of the individual responses. The response may be either the position or momentum of the oscillator. Our previous analysis can be applied directly to such systems.[4]

The x and y components of the equation of motion of charged particles in a uniform magnetic field [Eq. (2)] are formally identical with the equations for a one-dimensional harmonic oscillator $\dot{p}_x = -kx + F_x$, where F_x is the external force, and $\dot{x} = p_x/m$. Here the x coordinate replaces the y component of the velocity and there is no "force term" in the second equation. As before, we can combine the two real equations into a single complex equation,

$$(d\hat{p}/dt) - i\Omega\hat{p} = \hat{F}, \tag{20}$$

where $\hat{p} = p_x + \mathrm{i}m\Omega x$, $\hat{F} = F_x + i0$, and $\Omega = (k/m)^{1/2}$ is the harmonic oscillator frequency. The complex $\hat{p}$ plane is simply the phase plane, and the trajectory of a harmonic oscillator in this plane is a circle, traced out with angular velocity Ω. As before, it is convenient to transform to a *rotating phase space* which rotates with an angular velocity equal to the frequency ω of the driving force. This removes most of the rotation from the trajectories, since $\Omega \cong \omega$. Since the macroscopic response of the system of oscillators consists of a (weighted) sum (or integral) of the individual oscillator responses, we are interested in either the real or imaginary part of a quantity,

$$\hat{P} = N \int \hat{p}(t, \Omega') G(\Omega') d\Omega', \tag{21}$$

where $G(\Omega')$ is the fraction of the oscillators per unit frequency range. Equation (21) is formally identical with Eq. (7). Thus our previous results concerning cyclotron echoes in gyrating charged particles in a magnetic field are much more general than previously supposed. Figure 3 is now to be interpreted as a series of rotating phase plane (coordinates p_x, $m\Omega x$) diagrams for the response of the system of oscillators.

Our previous results now allow us to state the following sufficient conditions for two-pulse echo production:

(1) *a system of (almost) harmonic oscillators with a distribution of oscillator frequencies*. The spread in oscillator frequencies $\langle \Delta\Omega^2 \rangle^{1/2}$ must exceed $1/\tau$ in order for the direct response to the pulses not to mask the echo;

(2) *sufficient oscillator "lifetime"*. The collision frequency ν for phase-destroying collisions must be less than $1/\tau$ or equal to τ so that the latter do not eliminate the echoes.

(3) *one of several possible nonlinear effects*, to spoil the cancellation in Eq. (21), such as

(a) energy-dependent driving force,

(b) energy-dependent frequency (anharmonic oscillators), and

(c) energy-dependent lifetime (or relaxation process).

In Table I we list the types of echoes observed

TABLE I. Types of echoes.

Type	System	Probable nonlinearity	Refs.
Spin echo	*Precessing* nuclear spins in a magnetic field, electron spins in ferromagnetic and ferromagnetic materials	(a) (b)	1, 2, 16, 17
Photon echo	*Oscillating* electric dipoles of chromium ions in ruby crystal and of SF_6	(a)	18–20
Cyclotron echo	*Gyrating* free electrons of a plasma in a magnetic field	(b) (c)	3–15
Molecular echo	*Rotating* molecules in a gas *oscillating* electric dipoles in NH_3	(a)	21, 22
Plasma wave echo	*Streaming* free electrons in a plasma	(a) (b)	23–28
Fluxoid echo	*Fluxoid* excitations in type II superconductors	(b)	29, 30

so far, together with the nature of the oscillator systems and probable nonlinear mechanism.[15–30]

Since spin echoes have been known and understood for years, and spin resonance is generally thought of as a linear phenomenon, it is of considerable importance to discuss spin resonance in this new framework, and to point out clearly in what sense it is a nonlinear phenomenon. Discussions of the spin-resonance excitation of an elementary spin are frequently couched in terms of the complex amplitudes to be in the various spin states.[31] The applied radio-frequency magnetic field is treated as a perturbation which couples the various amplitudes. The set of equations governing these *amplitudes* is linear in each of the amplitudes and *linear in the perturbing rf field*, and therefore generally readily solved. However, from a physical standpoint, the quantities of final interest are not the amplitudes but rather some quadratic functionals of the various amplitudes. In particular, one is interested in the expectation value of the x, y, and z components of the spin (or magnetic moment **m**). One can, of course, compute the latter from the amplitudes, together with the appropriate matrix elements. It is generally simpler, however, to discuss the behavior of the magnetic moment **m** itself using the equation of motion[2,32,33]

$$(d\mathbf{m}/dt)=\gamma\mathbf{m}\times\mathbf{H}, \tag{22}$$

which follows from the same quantum mechanical equations (γ is the gyromagnetic ratio and H is the magnetic field.) This equation is inherently nonlinear and, as the following arguments show, the nonlinearity is of the energy-dependent driving force type.

It is convenient to resolve **m** and **H** into their components parallel to and perpendicular to the static magnetic field, i.e., $\mathbf{m}=\mathbf{m}_{||}+\mathbf{m}_{\perp}$ and $\mathbf{H}=\mathbf{H}_{||}+\mathbf{H}_{\perp}$. In magnetic resonance and spin echo experiments generally, $\mathbf{m}_{\perp}$ is the oscillating quantity of interest, $\mathbf{H}_{||}$ is the static magnetic field and $\mathbf{H}_{\perp}$ is the applied rf magnetic field. The perpendicular part of Eq. (22) may be written

$$(d\mathbf{m}_{\perp}/dt)+\boldsymbol{\omega}_s\times\mathbf{m}_{\perp}=\gamma\mathbf{m}_{||}\times\mathbf{H}_{\perp}, \tag{23}$$

where $\boldsymbol{\omega}_s=\gamma\mathbf{H}_{||}$ is the (vector) free-spin precession frequency, which depends upon the static magnetic field $\mathbf{H}_{||}$. We note the formal similarity of Eq. (23) and Eq. (2). In the absence of a driving force ($\gamma\mathbf{m}_{||}\times\mathbf{H}_{\perp}=0$), $\mathbf{m}_{\perp}$ traces out a circle with angular velocity ω_s in the $m_{\perp}$ plane. Note, however, that $m_{||}$ can be written as $(m^2-m_{\perp}^2)^{1/2}$ since Eq. (22) implies that $m^2=m_{||}^2+m_{\perp}^2$ is a constant. Thus Eq. (23) can also be written

$$(d\mathbf{m}/dt)+\boldsymbol{\omega}_s\times\mathbf{m}_{\perp}=\gamma(m^2-m_{\perp}^2)^{1/2}\mathbf{e}_{||}\times\mathbf{H}_{\perp}, \tag{24}$$

where $\mathbf{e}_{||}$ is a unit vector parallel to the static magnetic field.

It is now obvious from Eq. (24) that the effectiveness of driving force $\mathbf{H}_{\perp}$ diminishes with increasing $m_{\perp}$, and if we regard $m_{\perp}^2$ as a measure of the energy of excitation of the precessing spin, the *driving force is energy-dependent.* Thus spin systems fall into our previous framework.

It is useful to note that *most* resonantly-excited, two-state, quantum-mechanical systems can be described by a vector equation[34] of a form similar

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to Eq. (22)

$$(d\mathbf{r}/dt) = \boldsymbol{\omega} \times \mathbf{r}. \tag{25}$$

The physical interpretation of $\boldsymbol{\omega}$ and $\mathbf{r}$ depends on the particular system; but in terms of wave function, which may be written in terms of the amplitudes $a(t)$ and $b(t)$, to be in state a and b, respectively, $[\psi(t) = a(t)\psi_a + b(t)\psi_b]$ then

$$\begin{aligned} r_1 &= ab^* + a^*b & \omega_1 &= (V_{ab} + V_{ba})/\hbar \\ r_2 &= i(ab^* - a^*b) & \omega_2 &= i(V_{ab} - V_{ba})/\hbar \\ r_3 &= aa^* - bb^* & \omega_3 &= (E_a - E_b)/\hbar, \end{aligned} \tag{26}$$

are the x, y, and z components of the vectors $\mathbf{r}$ and $\boldsymbol{\omega}$. For the cases of common interest, where either electric or magnetic dipole transitions $(\Delta m = \pm 1)$ occur between the two states, certain components of $\mathbf{r}$ and $\boldsymbol{\omega}$ have simple physical interpretations. r_1 and r_2 are then proportional to the x and y components of the expected value (electric or magnetic) dipole moments, and ω_1 and ω_2 are proportional to the x and y components of the perturbing (electric or magnetic) field. ω_3 is the transition frequency, and thus $\hbar\omega_3$ is the transition energy. Since many two-state quantum mechanical transitions are described by Eq. (25), which is *identical in form* with Eq. (22), many of the concepts and methods of analysis developed in connection with spin resonance can be carried over to a larger class of systems. In particular, echo phenomena are to be expected in many two-state quantum-mechanical systems (or systems in which only transitions between two states are important). The photon echo[18–20] and molecular echo[21,22] in gases (Table I) are examples.

V. DISCUSSION

Two effects, neglected in the treatment of Sec. I, are sometimes of considerable importance, (a) electron motion along field lines because of their "thermal speeds" or temperature, and (b) the collective interaction between electrons due to "electron-generated" macroscopic electric fields. We discuss each briefly in turn.

In Sec. I we assumed that not only did the electrons have a distribution of cyclotron frequencies by virtue of the spatial inhomogeneity of the magnetic field and the spatial distribution of electrons, but also, that the cyclotron frequency of each individual electron did not change significantly during the time interval of interest. However, the electrons of a 0.1 eV (1160°K) plasma have a mean speed of $\sim 2 \times 10^7$ cm/sec and most electrons will have a significant fraction of this speed directed *along* a magnetic field line. Thus electrons may move a few centimeters along the field between the two pulses of a two-pulse echo and a few more centimeters between successive echoes. If, as is generally the case, the magnetic field strength varies along a field line, as well as from line to line, the cyclotron frequency changes slowly as the particle moves.[35] Our argument for the phase space regrouping (Fig. 3) and two-pulse echo production depends critically on the assumption that the electron gyrates through the same angle during subsequent intervals as if during the first interval τ or, in the case of an energy-dependent cyclotron frequency, the *change* in angle depends only upon the energy. We may estimate that the dephasing due to motion along the lines becomes important when the resulting accumulated phase change during the interval τ approaches on radian, i.e., when

$$v_z(\partial\omega_c/\partial_z)\tau \sim 1. \tag{27}$$

When the z-directed velocity distribution is Maxwell–Boltzmann, this leads to decay in the strength of the first echo by a factor[5,6]

$$F = \exp\{-[\tfrac{1}{4}(\partial\omega_c/\partial z)\tau^2]^2 2\kappa T/m\}. \tag{28}$$

For fields with a high degree of symmetry, the echo decay due to electron motion along the lines may not be as great as given by Eq. (28).

We have also assumed that electrons did not move out of the volume illuminated by the microwave field, and the fact that the more energetic electrons may do so is still an additional source of echo degradation. In fact, one might wonder why three-pulse echoes are not almost completely destroyed by the motion of the electrons along magnetic field lines to regions outside of the microwave waveguide [Fig. 1(b)]. We have previously noted that electrons typically acquire several electron volts of energy perpendicular to the field from the rf pulses, then this energy is shared with the parallel direction after a few collisions. Thus, most electrons are able

[35] A notable exception is the "toroidal" field of a straight wire in which $B\phi = \mu_0 I/2\pi\rho$ varies from line to line but not along a given line.

to leave one microwave volume very quickly. However, this probably leads simply to a rapid mixing of the electrons which are inside the microwave volume with those which are outside since all but a very few (the most energetic) electrons are prevented from leaving the plasma region by the sheath which forms at the plasma boundaries. A certain fraction of these electrons given roughly by the ratio of microwave volume to plasma volume will have found their way *back* into the microwave volume after repeated reflections from the sheath.

In Sec. I we also neglected the electric fields generated by the charges themselves, i.e., the collective effects. From the discussion of Sec. IV, we might say it is simply a matter of finding the normal modes of the system, including macroscopic electric fields of the charges themselves, together with the nonlinear interaction between these modes. This is a formidable task, in general, but can actually be carried out for at least one simple case:[36] a zero temperature, plane-layered slab model. It is found that each layer (of constant electron density and magnetic field) oscillates independently of the others with frequency

$$\omega_H(\omega_c^2+\omega_p^2)^{1/2}, \tag{29}$$

where $\omega_p=(ne^2/\epsilon_0 m)^{1/2}$ is the local electron plasma frequency. ω_H is generally referred to as the upper hybrid frequency. It is no longer necessary to require a spatially inhomogeneous magnetic field, since according to Eq. (29) a variation in electron density from layer to layer also gives rise to a distribution of oscillator frequencies. Equation (8) (with ω'_H replacing ω_c') tells us that the single-pulse decay time is related to the range in upper hybrid frequencies, and hence the electron density. It has been confirmed experimentally in an afterglow plasma that the duration of the single-pulse transient response increases in the expected manner as the electron density decays.[37]

It has also been shown theoretically that when *large amplitude* oscillations of the layered-slab model are considered, the oscillation frequency of each layer departs from that given by Eq. (29) in a manner which is amplitude dependent, i.e., the individual layers may be thought of as a collection of *anharmonic* oscillators. Thus the layered-slab model provides both a distribution of oscillator frequencies and a nonlinearity of the energy-dependent frequency type.[36] Both features of the model can be traced directly to the existence of gradients in electron density. Recent experiments show that this model appears to have substantial relevance to echo production in plasmas.[8,38,39]

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